

ON THE UPSILON INVARIANT OF CABLE KNOTS

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Abstract

In this paper, we study the behavior of $\Upsilon_K(t)$ under the cabling operation, where $\Upsilon_K(t)$ is the knot concordance invariant defined by Ozsváth, Stipsicz, and Szabó, associated to a knot $K \subset S^3$. The main result is an inequality relating $\Upsilon_K(t)$ and $\Upsilon_{K_{p,q}}(t)$, which generalizes the inequalities of Hedden [6] and Van Cott [22] on the Ozsváth-Szabó τ -invariant. As applications, we give a computation of $\Upsilon_{(T_{2,-3})_{2,2n+1}}(t)$ for $n \geq 8$, and we also show that the set of iterated $(p, 1)$ -cables of $Wh^+(T_{2,3})$ for any $p \geq 2$ span an infinite-rank summand of topologically slice knots.

1 Introduction

The complete knot Floer chain complex $CFK^\infty(K)$ is a bifiltered, Maslov graded chain complex associated to a knot $K \subset S^3$, introduced by Ozsváth and Szabó [18], and independently by Rasmussen [21]. *A priori*, the bifiltered chain homotopy type of $CFK^\infty(K)$ is an isotopy invariant of the knot K . By exploiting the TQFT-like aspects of the theory, however, it turns out that $CFK^\infty(K)$ also contains a lot of interesting information about the concordance class of K , see [9] for a survey. One typical example is the Ozsváth-Szabó τ -invariant, which came to stage relatively early and attracted a lot of attention [15]. Roughly speaking, $\tau(K)$ is a concordance homomorphism that takes its values in \mathbb{Z} and is defined by examining $\widehat{CFK}(K)$, which is a small portion of $CFK^\infty(K)$. Moreover, τ provides a lower bound to the smooth four-genus of a knot i.e. $|\tau(K)| \leq g_4(K)$. Among many applications, Ozsváth and Szabó showed that τ can be used to resolve a conjecture of Milnor first proven by Kronheimer and Mrowka using gauge theory [12], that $g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$, where $T_{p,q}$ is the (p, q) -torus knot.

Recently, by using more information from $CFK^\infty(K)$, Ozsváth, Stipsicz, and Szabó introduced a more powerful concordance invariant that generalizes τ [19]. This invariant takes the form of a homomorphism from the smooth knot concordance group \mathcal{C} to the group of piecewise linear functions on $[0, 2]$. Thus, for every knot $K \subset S^3$, they associate a piecewise linear function $\Upsilon_K(t)$, where $t \in [0, 2]$ that depends only on the concordance type of K .

Besides being a concordance homomorphism, $\Upsilon_K(t)$ also enjoys many other nice properties, some of which we list below.

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- (1) (Symmetry) $\Upsilon_K(t) = \Upsilon_K(2 - t)$,
- (2) (4-genus bound) $|\Upsilon_K(t)| \leq tg_4(K)$ for $0 \leq t \leq 1$,
- (3) (Recovers τ) The slope of $\Upsilon_K(t)$ at $t = 0$ is $-\tau(K)$.

In this paper we study how Υ behaves under the cabling operation. Such results for τ can be found in [3, 4, 6, 8, 20, 22], among which we restate two results. The first one is due to Hedden, obtained by carefully comparing the knot Floer chain complex of a knot and that of its cable.

Theorem 1.1. ([6]) Let $K \subset S^3$ be a knot, and $p > 0$, $n \in \mathbb{Z}$. Then

$$p\tau(K) + \frac{pn(p-1)}{2} \leq \tau(K_{p,pn+1}) \leq p\tau(K) + \frac{(pn+2)(p-1)}{2},$$

Later Van Cott used the genus bound and homomorphism property satisfied by τ , together with nice constructions of cobordism between cable knots to extend the above inequality to (p, q) -cables.

Theorem 1.2. ([22]) Let $K \subset S^3$ be a knot, and (p, q) be a pair of relatively prime numbers such that $p > 0$. Then

$$p\tau(K) + \frac{(p-1)(q-1)}{2} \leq \tau(K_{p,q}) \leq p\tau(K) + \frac{(p-1)(q+1)}{2},$$

In view of the success in understanding the effect of knot cabling on the τ -invariant, it is natural to wonder what happens to Υ . In this paper we show that a portion of Υ behaves very similarly to τ . Indeed, adapting the strategies of Hedden and Van Cott on studying τ to the context of Υ , we can prove the following result:

Theorem 1.3. Let $K \subset S^3$ be a knot, and (p, q) be a pair of relatively prime numbers such that $p > 0$. Then

$$\Upsilon_K(pt) - \frac{(p-1)(q+1)t}{2} \leq \Upsilon_{K_{p,q}}(t) \leq \Upsilon_K(pt) - \frac{(p-1)(q-1)t}{2},$$

when $0 \leq t \leq \frac{2}{p}$.

Note that by differentiating the above inequality at $t = 0$, we recover Theorem 1.2. One also easily sees this inequality is sharp by examining the case when K is the unknot: when $q > 0$, then the upper bound is achieved, and when $q < 0$ the lower bound is achieved. However, when $p > 2$ the behavior of $\Upsilon_{K_{p,q}}(t)$ for $t \in [\frac{2}{p}, 2 - \frac{2}{p}]$ is still unknown to the author.

Theorem 1.3 can often be used to determine the Υ function of cables with limited knowledge of their complete knot Floer chain complexes. For an example, we show how our theorem can be used to deduce $\Upsilon_{(T_{2,-3})_{2,2n+1}}(t)$ for $n \geq 8$; these examples are, on the face of it, rather difficult to compute, since none of

them is an L -space knot. Despite this, armed only with Theorem 1.3 and the knot Floer homology groups of the knots in this family, we are able to obtain complete knowledge of Υ .

Perhaps more striking, however, Theorem 1.3 can be used to easily show that certain subsets of the smooth concordance group freely generate infinite-rank summands. To this end, let $D = Wh^+(T_{2,3})$ be the untwisted positive whitehead double of the trefoil knot and let $J_n = ((D_{p,1})\dots)_{p,1}$ denote the n -fold iterated $(p, 1)$ -cable of D for some fixed $p > 1$ and some positive integer n . Theorem 1.3, together with general properties of Υ , yield the following

Corollary 1.4. The family of knots J_n for $n = 1, 2, 3, \dots$ are linearly independent in \mathcal{C} and span an infinite-rank summand consisting of topologically slice knots.

To the best of the author's knowledge, Corollary 1.4 provides the first satellite operator on the smooth concordance group of topologically slice knots whose iterates (for a fixed knot) are known to be independent; moreover, in this case they are a summand. Note J_n has trivial Alexander polynomial. The first known example of infinite-rank summand of knots with trivial Alexander polynomial is generated by $D_{n,1}$ for $n \in \mathbb{Z}^+$, due to Kim and Park [11]. Their example, however, like the families of topologically slice knots studied by [19], involved rather non-trivial calculations of Υ (for instance, those of [19] involved rather technical calculations with the bordered Floer invariants). The utility of Theorem 1.3 is highlighted by the ease with which the above family is handled. We also refer the interested reader to [1] for a host of other applications of Theorem 1.3 to the study of the knot concordance group.

To conclude the introduction, it is worth mentioning that Hom achieved a complete understanding of the behavior of τ under cabling by introducing the ϵ -invariant [8]. In particular, $\tau(K_{p,q})$ is always one of the two bounds appearing in Theorem 1.2, depending on the value of $\epsilon(K)$. However, in the context of Υ the story is not true, even for L -space knots whose knot Floer chain complexes are relatively simple. For instance, $\Upsilon_{(T_{2,3})_{n,2n-1}}(t)$ is computed in [19] but it does not equal either bound appearing in Theorem 1.3. This suggests the behavior of Υ under cabling is more complicated. For example, it would be interesting if one can find suitable auxiliary invariants serving a similar role as ϵ -invariant.

Outline. The organization of the rest of the paper is as following: in Section 2, we review the definition of Υ . In Section 3, we prove our main theorem. In Section 4 we give two applications of Theorem 1.3.

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2 Preliminaries

We work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout the entire paper. We also assume that the reader is familiar with the basic setup of knot Floer homology. For more details, see [18, 21].

In this section, we will briefly review the construction of $\Upsilon_K(t)$ of a given knot K , setting up some notations at the same time. The original definition of $\Upsilon_K(t)$ is based on a t -modified knot Floer chain complex, see [19]. Shortly thereafter, Livingston reformulated $\Upsilon_K(t)$ in terms of the complete knot Floer chain complex $CFK^\infty(K)$. We find it convenient to work with Livingston's definition, which we recall below.

Denote $CFK^\infty(K)$ by $C(K)$ for convenience. Note that $C(K)$ comes with a $\mathbb{Z} \oplus \mathbb{Z}$ -filtration, namely the Alexander filtration and the algebraic filtration. Actually, to be more precise, $C(K)$ is only well defined up to bifiltered chain homotopy equivalence, unless we fix some compatible Heegaard diagram for K and some auxiliary data.

Now for any $t \in [0, 2]$, one can define a filtration on $C(K)$ as follows. First, define a real-valued (grading) function on $C(K)$ by

$$\mathcal{F}_t = \frac{t}{2} Alex + (1 - \frac{t}{2}) Alg,$$

which is a convex linear combination of Alexander and algebraic gradings. Associated to this function, one can construct a filtration given by $(C(K), \mathcal{F}_t)_s = (\mathcal{F}_t)^{-1}(-\infty, s]$. It is easy to see that the filtration induced by \mathcal{F}_t is compatible with the differential of $C(K)$, i.e. $\mathcal{F}_t(\partial x) \leq \mathcal{F}_t(x)$, $\forall x \in C(K)$. Let

$$\nu(C(K), \mathcal{F}_t) = \min\{s \in \mathbb{R} \mid H_0((C(K), \mathcal{F}_t)_s) \rightarrow H_0(C(K)) \text{ is nontrivial}\}.$$

Here H_0 stands for the homology group with Maslov grading 0. With these preparations, Υ is defined as following.

Definition 2.1. $\Upsilon_K(t) = -2\nu(C(K), \mathcal{F}_t)$.

It is proven in [13] that the above definition of $\Upsilon_K(t)$ is equivalent to the one given by Ozsváth, Stipsicz, and Szabó in [19].

3 Proof of the main theorem

The proof of Theorem 1.3 will be divided into two parts: in Subsection 3.1 we will prove the inequality for the $(p, pn+1)$ cable of a knot by adapting Hedden's strategy in [3, 6], and then in Subsection 3.2 we will upgrade the inequality to cover the (p, q) -cable of a knot by applying Van Cott's argument in [22].

3.1 Upsilon of $(p, pn + 1)$ -cable

Following [3, 4, 6], we will begin with introducing a nice Heegaard diagram which encodes both the original knot K and its cable $K_{p,pn+1}$.

For any knot $K \subset S^3$, there exists a compatible Heegaard diagram $H = (\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_{g-1}, \mu\}, w, z)$. Moreover, by stabilizing we can assume μ to be the meridian of the knot K and that it only intersects α_g , and there is a 0-framed longitude of the knot K on Σ which does not intersect α_g . From now on, we will always assume that the Heegaard diagram H for K satisfies all these properties.

Let $H = (\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_{g-1}, \mu\}, w, z)$ be a Heegaard diagram for the knot K as above. By modifying μ and adding an extra base point z' , we can construct a new Heegaard diagram with three base points $H(p, n) = (\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_{g-1}, \tilde{\beta}\}, w, z, z')$. More precisely, $\tilde{\beta}$ is obtained by winding μ along an n -framed longitude $(p-1)$ times, and the new base point z' is placed at the tip of the winding region such that the arc δ' connecting w and z' has intersection number p with $\tilde{\beta}$. Note $\tilde{\beta}$ can be deformed to μ through an isotopy that does not cross the base points $\{w, z\}$. See Figure 1 and Figure 2 for an example. The power of $H(p, n)$ lies in the fact that it specifies both K and $K_{p,pn+1}$ at the same time, as pointed out by the following lemma.

Lemma 3.1. (Lemma 2.2 of [3]) Let $H(p, n)$ be a Heegaard diagram described as above. Then

1. Ignoring z' , we get a doubly-pointed diagram $H(p, n, w, z)$ which specifies K .
2. Ignoring z , we get a doubly-pointed diagram $H(p, n, w, z')$ which specifies the cable knot $K_{p,pn+1}$.

This implies that the two knot Floer chain complexes $CFK^\infty(H(p, n, w, z))$ and $CFK^\infty(H(p, n, w, z'))$ are closely related. More precisely, by forgetting the Alexander filtrations, both $CFK^\infty(H(p, n, w, z))$ and $CFK^\infty(H(p, n, w, z'))$ are isomorphic to $CF^\infty(H(p, n, w))$. Therefore, in order to get a more transparent correspondence between these two complexes, we will compare the Alexander gradings of the intersection points with respect to the two different base points z and z' .

For the sake of a clearer discussion, we fix some notation and terminology to deal with the intersection points. For convenience, we assume $n \geq 0$ through out the discussion and remark that the case when $n < 0$ can be handled in a similar way. Note $\tilde{\beta}$ intersects α_g at $2(p-1)n + 1$ points, and we label them as $x_0, \dots, x_{2(p-1)n}$, starting at the out-most layer from left to right, and then the second layer from left to right, and so on. On the other hand, $\tilde{\beta}$ could also intersect other α -curves besides α_g , and we label these points by $y_0^{(k)}, \dots, y_{2(p-1)-1}^{(k)}$. Here k enumerates the intersections of the n -framed longitude with α_i , $i \neq g$, and the order of this enumeration is irrelevant. The lower index is again ordered following a layer by layer convention, from outside to inside,

but we require that $y_0^{(k)}$ can be connected to x_{2n} by an arc on $\tilde{\beta}$ which neither intersects δ nor δ' , the short arcs connecting the base points. See Figure 2 for an example. The generators will be partitioned into p classes: all the generators of the form $\{x_{2i}, \mathbf{a}\}$ or $\{y_{2i}^{(k)}, \mathbf{b}\}$ will be called *even intersection points* or *0-intersection points*, and *odd intersection points* otherwise; odd generators of the form $\{x_{2i+1}, \mathbf{a}\}$ or $\{y_{2\lceil \frac{i+1}{n} \rceil - 1}^{(k)}, \mathbf{b}\}$ will be called $(p - \lceil \frac{i+1}{n} \rceil)$ -*intersection points*. Here \mathbf{a}, \mathbf{b} are $(g-1)$ -tuple in $Sym^{g-1}(\Sigma)$. Note that essentially we are classifying odd intersection points into $(p-1)$ classes by the following principle: if its $\tilde{\beta}$ -component sits on the i -th layer (we count the layers from outside to inside), then it is called a $(p-i)$ -intersection point.

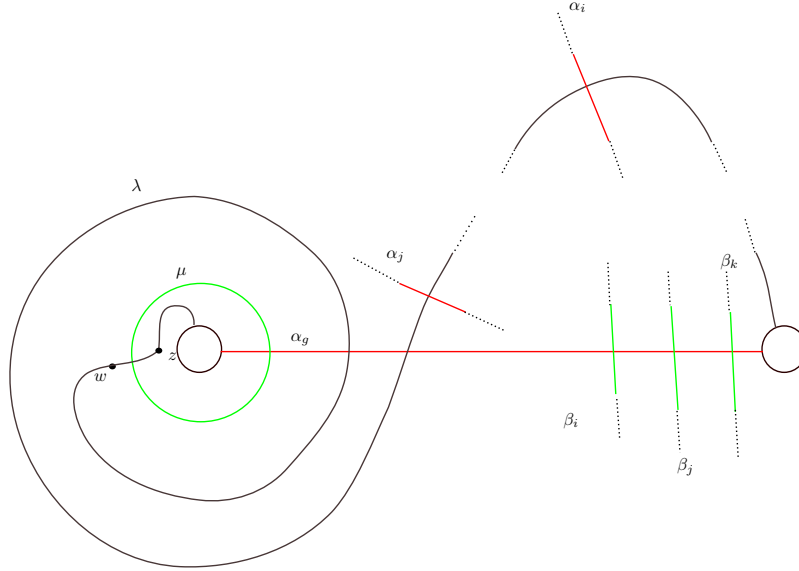


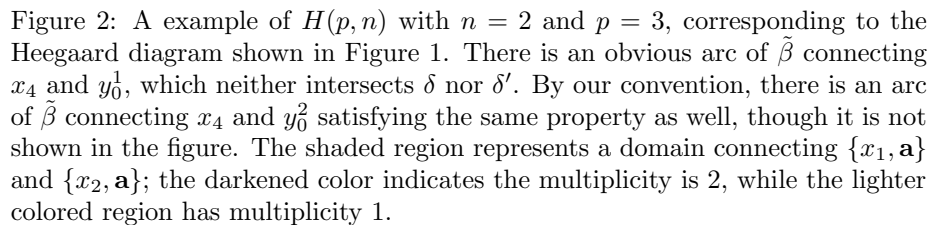
Figure 1: A compatible Heegaard diagram H for K . λ is a 2-framed longitude, and according to our assumption that the 0-framed longitude can be chosen not to hit α_g , λ can be chosen to intersect α_g twice.

We denote the Alexander grading by A (by A') when we use the base point z (base point z'). The comparison of Alexander filtrations is summarized in the following proposition.

Proposition 3.2. With the choice of Heegaard diagrams as described above and let \mathbf{x} be an l -intersection point, where $l \in \{0, 1, \dots, p-1\}$, then

$$A'(\mathbf{x}) = pA(\mathbf{x}) + \frac{pn(p-1)}{2} + l.$$

Proposition 3.2 can be viewed as a generalization of the comparison used in [3, 6], in which only $\{x_i, \mathbf{a}\}$ for $i \leq n$ were shown to satisfy the above equation. In studying τ , having just a comparison for $\{x_i, \mathbf{a}\}$ for $i \leq n$ would



suffice: first, Hedden observed that in the case when $|n|$ is sufficiently large they account for the top Alexander graded generators of $\widehat{CFK}(K_{p,pn+1})$ that determine $\tau(K_{p,pn+1})$; second, the behavior of τ for small n can be deduced from the large- n case by using crossing change inequality of τ . In contrast, the lower Alexander graded elements of $CFK^\infty(K_{p,pn+1})$ may play a role in Υ , even though they do not affect τ . Therefore in the current paper we have to carry out a comparison for all types of generators. To accomplish this goal, we quote and extend some of the lemmas used in [3, 6] below, after which Proposition 3.2 will follow easily.

Lemma 3.3. When $1 \leq j \leq (p-1)n$, we have

$$A(\{x_{2j-1}, \mathbf{a}\}) - A(\{x_{2j}, \mathbf{a}\}) = 0 \quad (3.1)$$

$$A'(\{x_{2j-1}, \mathbf{a}\}) - A'(\{x_{2j}, \mathbf{a}\}) = p - \lceil \frac{j}{n} \rceil \quad (3.2)$$

For an arbitrary k , when $0 \leq i \leq (p-2)$, we have

$$A(\{y_{2i+1}^{(k)}, \mathbf{a}\}) - A(\{y_{2i}^{(k)}, \mathbf{a}\}) = 0 \quad (3.3)$$

$$A'(\{y_{2i+1}^{(k)}, \mathbf{a}\}) - A'(\{y_{2i}^{(k)}, \mathbf{a}\}) = p - (i+1) \quad (3.4)$$

Proof Note that there is a Whitney disk ϕ connecting $\{x_{2j-1}, \mathbf{a}\}$ to $\{x_{2j}, \mathbf{a}\}$ (See Figure 2). It is the product of a constant map in $Sym^{g-1}(\Sigma)$ and the map represented by the disk which connects x_{2j-1} and x_{2j} , with boundary consisting of a short arc of α_g and an arc of $\tilde{\beta}$ that spirals into the winding region $p - \lceil \frac{j}{n} \rceil$ times and then makes a turn out. We can see that $n_w(\phi) = n_z(\phi) = 0$ and $n_{z'}(\phi) = p - \lceil \frac{j}{n} \rceil$. Therefore, $A(\{x_{2j-1}, \mathbf{a}\}) - A(\{x_{2j}, \mathbf{a}\}) = n_z(\phi) - n_w(\phi) = 0$ and $A'(\{x_{2j-1}, \mathbf{a}\}) - A'(\{x_{2j}, \mathbf{a}\}) = n_{z'}(\phi) - n_w(\phi) = p - \lceil \frac{j}{n} \rceil$. We have obtained equation (3.1) and (3.2). The proof for (3.3) and (3.4) will follow a similar line, and hence is omitted. \square

Lemma 3.4. When $0 \leq j \leq (p-1)n$, we have

$$p(A(\{x_0, \mathbf{a}\}) - A(\{x_{2j}, \mathbf{a}\})) = A'(\{x_0, \mathbf{a}\}) - A'(\{x_{2j}, \mathbf{a}\}) \quad (3.5)$$

For an arbitrary k , when $0 \leq i \leq (p-2)$, we have

$$p(A(\{y_0^{(k)}, \mathbf{b}\}) - A(\{y_{2i}^{(k)}, \mathbf{b}\})) = A'(\{y_0^{(k)}, \mathbf{b}\}) - A'(\{y_{2i}^{(k)}, \mathbf{b}\}) \quad (3.6)$$

$$p(A(\{x_{2n}, \mathbf{a}\}) - A(\{y_0^{(k)}, \mathbf{b}\})) = A'(\{x_{2n}, \mathbf{a}\}) - A'(\{y_0^{(k)}, \mathbf{b}\}). \quad (3.7)$$

Proof First we prove Equation (3.5). Note that $\epsilon(\{x_{2j}, \mathbf{a}\}, \{x_0, \mathbf{a}\})$ can be represented by a curve γ on Σ , which is obtained by first connecting x_{2j} to x_0 along α_g , and then by an arc on $\tilde{\beta}$ which starts from x_0 and winds j times counterclockwise to arrive at x_{2j} (Figure 3). Note that $[\epsilon(\{x_{2j}, \mathbf{a}\}, \{x_0, \mathbf{a}\})] = 0 \in H_1(S^3, \mathbb{Z})$, hence $[\gamma] = \Sigma l_i \alpha_i + \Sigma k_i \beta_i$, with β_g is viewed as $\tilde{\beta}$. Let $c = \gamma - \Sigma l_i \alpha_i - \Sigma k_i \beta_i$, then c bounds a domain on Σ . Note that since $\delta \cdot \gamma = \delta' \cdot \gamma = 0$, we have $\delta' \cdot c = \delta' \cdot (-k_g \tilde{\beta}) = -k_g p = p(\delta \cdot (-k_g \tilde{\beta})) = p(\delta \cdot c)$, where “ \cdot ” stands for the intersection number. Equation (3.5) follows.

The proofs for the other two equations follow a similar line. Note the key point in the above argument is that the ϵ -class of the two generators can be

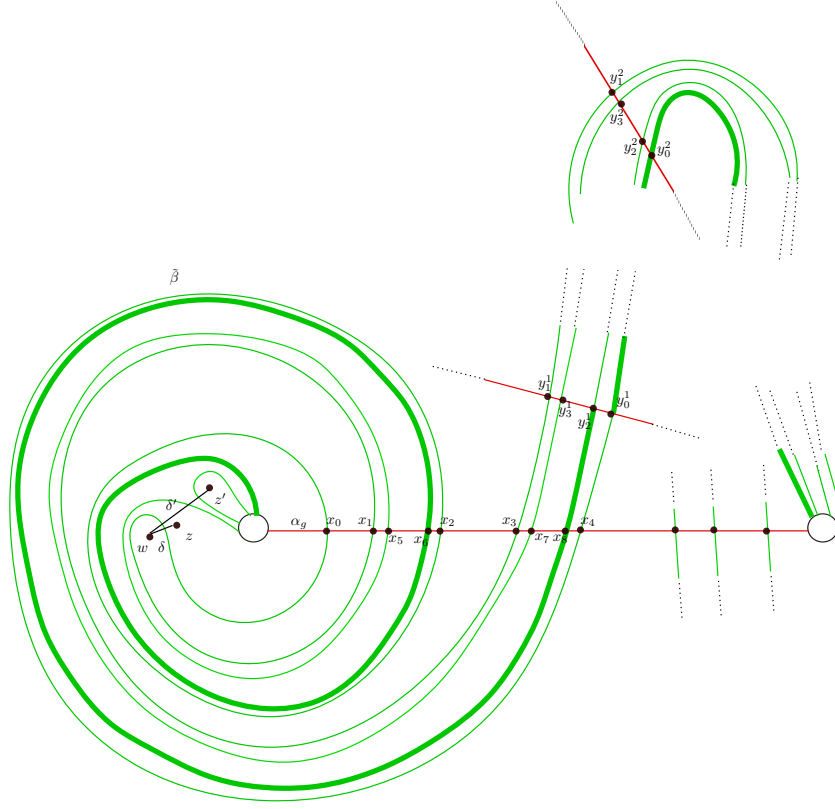


Figure 4: The thickened curve is an arc on $\tilde{\beta}$ connecting $y_0^{(1)}$ and $y_2^{(1)}$ that does not intersect δ nor δ' .

Lemma 2.5 of [6], $A'(\{x_0, \mathbf{a}\}) = pA(\{x_0, \mathbf{a}\}) + \frac{pn(p-1)}{2}$. Note that for any other intersection point \mathbf{u} , as long as $A'(\{x_0, \mathbf{a}\}) - A'(\mathbf{u}) = p(A(\{x_0, \mathbf{a}\}) - A(\mathbf{u}))$, then we have $A'(\mathbf{u}) = pA(\mathbf{u}) + \frac{pn(p-1)}{2}$ as well. Now the case when $l = 0$ (even intersection points) follows easily from this observation, Lemma 3.4, and Lemma 3.5. The other cases is then an easy consequence of the $l = 0$ case and Lemma 3.3. \square

Let $C = CF^\infty(H(p, n, w))$ be the chain complex obtained by forgetting the Alexander filtration. And let $(\mathcal{F}_t) = \frac{t}{2}A + (1 - \frac{t}{2})Alg$, and $(\mathcal{F}_t)' = \frac{t}{2}A' + (1 - \frac{t}{2})Alg$ be two grading functions on C defined by using the two Alexander gradings A and A' corresponding to z and z' respectively. Then the filtrations corresponding to (\mathcal{F}_t) and $(\mathcal{F}_t)'$ satisfy the following relation.

Lemma 3.6. For $p, n \in \mathbb{Z}$, $p > 0$, and $0 \leq t \leq \frac{2}{p}$, we have

$$(C, \mathcal{F}_t')_{s + \frac{pn(p-1)t}{4}} \subset (C, \mathcal{F}_{pt})_s \subset (C, \mathcal{F}_t')_{s + \frac{(pn+2)(p-1)t}{4}}.$$

Proof Let \mathbf{x} be a generator of C . Assume $U^{-k}\mathbf{x} \in (C, \mathcal{F}_{pt})_s$, then

$$\frac{pt}{2}(A(\mathbf{x}) + k) + (1 - \frac{pt}{2})k = \frac{pt}{2}A(\mathbf{x}) + k \leq s.$$

Combine the above inequality with Prop. 3.2, we have

$$\begin{aligned} & \frac{t}{2}(A'(\mathbf{x}) + k) + (1 - \frac{t}{2})k \\ & \leq \frac{t}{2}(pA(\mathbf{x}) + \frac{pn(p-1)}{2} + p - 1 + k) + (1 - \frac{t}{2})k \\ & = \frac{pt}{2}A(\mathbf{x}) + k + \frac{t}{2}\frac{pn(p-1)}{2} + \frac{t}{2}(p-1) \\ & \leq s + \frac{(pn+2)(p-1)t}{4}. \end{aligned}$$

Hence $U^{-k}\mathbf{x} \in (C, \mathcal{F}'_t)_{s + \frac{(pn+2)(p-1)t}{4}}$, and therefore

$$(C, \mathcal{F}_{pt})_s \subset (C, \mathcal{F}'_t)_{s + \frac{(pn+2)(p-1)t}{4}}.$$

Similarly, if we assume $U^{-k}\mathbf{x} \notin (C, \mathcal{F}_{pt})_s$, then

$$\frac{pt}{2}(A(\mathbf{x}) + k) + (1 - \frac{pt}{2})k > s.$$

Again, in view of the above inequality and Prop. 3.2, we have

$$\begin{aligned} & \frac{t}{2}(A'(\mathbf{x}) + k) + (1 - \frac{t}{2})k \\ & \geq \frac{t}{2}(pA(\mathbf{x}) + \frac{pn(p-1)}{2} + k) + (1 - \frac{t}{2})k \\ & = \frac{pt}{2}A(\mathbf{x}) + k + \frac{t}{2}\frac{pn(p-1)}{2} \\ & > s + \frac{pn(p-1)t}{4} \end{aligned}$$

Hence $U^{-k}\mathbf{x} \notin (C, \mathcal{F}'_t)_{s + \frac{pn(p-1)t}{4}}$, and therefore

$$(C, \mathcal{F}'_t)_{s + \frac{pn(p-1)t}{4}} \subset (C, \mathcal{F}_{pt})_s.$$

□

Proof of Theorem 1.3 for $(p, pn+1)$ -cable. Recall that

$$\nu(C, \mathcal{F}_t) = \min\{s \mid H_0((C, \mathcal{F}_t)_s) \rightarrow H_0(C) \text{ is nontrivial}\},$$

and $\nu(C, \mathcal{F}'_t)$ is understood similarly. Now set $s = \nu(C, \mathcal{F}_{pt})$ in lemma 3.6, we have

$$\nu(C, \mathcal{F}_{pt}) + \frac{pn(p-1)t}{4} \leq \nu(C, \mathcal{F}'_t) \leq \nu(C, \mathcal{F}_{pt}) + \frac{(pn+2)(p-1)t}{4}.$$

Recall $\Upsilon_K(pt) = -2\nu(C, \mathcal{F}_{pt})$ and $\Upsilon_{K_{p,pn+1}}(t) = -2\nu(C, \mathcal{F}'_t)$, so by mutiplying -2 the above inequality translates to

$$\Upsilon_K(pt) - \frac{(pn+2)(p-1)t}{2} \leq \Upsilon_{K_{p,pn+1}}(t) \leq \Upsilon_K(pt) - \frac{pn(p-1)t}{2}.$$

□

3.2 Upsilon of (p, q) -cable

Denote the smooth knot concordance group by \mathcal{C} . Let $\theta : \mathcal{C} \rightarrow \mathbb{R}$ be a concordance homomorphism such that $|\theta(K)| \leq g_4(K)$ and $\theta(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ when $p, q > 0$. In [22], Van Cott proved that if we fix a knot K and $p > 0$, and let

$$h(l) = \theta(K_{p,l}) - \frac{(p-1)l}{2},$$

then we have

$$-(p-1) \leq h(n) - h(r) \leq 0, \quad (3.8)$$

when $n > r$ such that both n and r relatively prime to p .

Remark. The concordance homomorphism studied by Van Cott has range \mathbb{Z} rather than \mathbb{R} , but by checking the argument in [22], it is straightforward to see that this choice will not affect the inequality stated above.

Now note that fixing $t \in (0, 2/p]$, $\frac{-\Upsilon_K(t)}{t}$ is a concordance homomorphism which lower bounds the four-genus, and $\frac{-\Upsilon_{T_{p,q}}(t)}{t} = \frac{(p-1)(q-1)}{2}$ when $q > 0$ ([14]). So we can take θ to be $\frac{-\Upsilon_K(t)}{t}$ and apply inequality (3.8), from which we get

$$0 \leq \bar{h}(n, t) - \bar{h}(r, t) \leq (p-1)t, \quad (3.9)$$

where $\bar{h}(n, t) = \Upsilon_{K_{p,n}}(t) + \frac{(p-1)nt}{2}$. It is easy to see inequality (3.9) is true at $t = 0$ as well, and hence it holds for $0 \leq t \leq \frac{2}{p}$.

Following essentially the argument of Corollary 3 in [22], we conclude the proof of our main theorem as below:

Proof of Theorem 1.3 for (p, q) -cable. Recall $0 \leq t \leq \frac{2}{p}$. First we will show that

$$\Upsilon_{K_{p,q}}(t) \geq \Upsilon_K(pt) - \frac{(p-1)(q+1)t}{2}.$$

Take r to be any integer such that $q \geq pr + 1$, then by inequality (3.9) we have

$$\bar{h}(q, t) - \bar{h}(pr + 1, t) \geq 0.$$

In view of the definition of \bar{h} , the above inequality translates to

$$\Upsilon_{K_{p,q}}(t) \geq \Upsilon_{K_{p,pr+1}}(t) - \frac{(p-1)(q-pr-1)t}{2}. \quad (3.10)$$

From the previous subsection, we have

$$\Upsilon_{K_{p,pr+1}}(t) \geq \Upsilon_K(pt) - \frac{(pr+2)(p-1)t}{2}.$$

Combining this and inequality (3.10), we get

$$\Upsilon_{K_{p,q}}(t) \geq \Upsilon_K(pt) - \frac{(p-1)(q+1)t}{2}.$$

The other half of the inequality follows from an analogous argument by considering $\bar{h}(pl+1, t) - \bar{h}(q, t) \geq 0$, where l is an integer such that $q \leq pl+1$. We omit the details. \square

4 Applications

4.1 Computation of $\Upsilon_{(T_{2,-3})_{2,2n+1}}(t)$

In this subsection, we show how one can compute $\Upsilon_{(T_{2,-3})_{2,2n+1}}(t)$ by using our theorem together with $\widehat{HFK}((T_{2,-3})_{2,2n+1})$, for $n \geq 8$. Note none of these knots is an L-space knot. For easier illustration, we only give full procedure of the computation for the case $K = (T_{2,-3})_{2,17}$. The general case can be done in a similar way.

By Proposition 4.1 of [3], for $i \geq 0$, we have

$$\widehat{HFK}(K, i) \cong \begin{cases} \mathbb{F}_{(2)} & i = 10 \\ \mathbb{F}_{(1)} & i = 9 \\ \mathbb{F}_{(1)} \oplus \mathbb{F}_{(0)} & i = 8 \\ \mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)} & i = 7 \\ \mathbb{F}_{(i-8)} & 0 \leq i \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Here the subindex stands for the Maslov grading. Note that by using the symmetry $\widehat{HFK}_d(K, i) = \widehat{HFK}_{d-2i}(K, -i)$ [18], the above equation actually tells us the whole $\widehat{HFK}(K)$. Now thinking $CFK^\infty(K)$ as $\widehat{HFK}(K) \otimes \mathbb{F}[U, U^{-1}]$ when regarded as an $\mathbb{F}[U, U^{-1}]$ -module, we see the lattice points supporting generators with Maslov grading 0 are $(0, 7)$, $(7, 0)$, $(i, 8-i)$, where $-1 \leq i \leq 9$. Here, for example, $(0, 7)$ means the corresponding generator has algebraic grading 0 and Alexander grading 7.

Note by Theorem 1.2 of [6], $\tau(K) = 7$. In view of Theorem 13.1 in [13], we see that for $t \in [0, \epsilon]$, $\Upsilon_K(t) = -2s$, where $s = \frac{t}{2} \cdot 7 + (1 - \frac{t}{2}) \cdot 0 = \frac{7t}{2}$ and ϵ is sufficiently small. In other words, when t is small, the Υ_K is determined by the \mathcal{F}_t grading of the generator at $(0, 7)$. Now by Theorem 7.1 in [13], singularities of $\Upsilon_K(t)$ can only occur at time t when there is a line of slope $1 - \frac{2}{t}$ that contains at least two lattice points supporting generators of Maslov grading 0. The only

$t \in (0, 1)$ satisfying this property is $\frac{2}{3}$, giving a line of slope -2 that passes through the lattice points $(0, 7)$ and $(-1, 9)$.

So far, we can see that $\Upsilon_K(t)$ is either one of the two below, depending on whether $\frac{2}{3}$ is a singular point or not.

$$\Upsilon_K(t) = \begin{cases} -7t, & t \in [0, \frac{2}{3}] \\ 2 - 10t, & t \in [\frac{2}{3}, 1] \end{cases} \quad (4.1)$$

Or

$$\Upsilon_K(t) = -7t, \quad t \in [0, 1]. \quad (4.2)$$

Note $T_{2,-3}$ is alternating, so we can apply Theorem 1.14 in [19] to obtain $\Upsilon_{T_{2,-3}}(t) = 1 - |1 - t|$, when $t \in [0, 2]$. Applying Theorem 1.3 we see that when $\frac{1}{2} \leq t \leq 1$, we have

$$2 - 11t \leq \Upsilon_K(t) \leq 2 - 10t.$$

Now we see only (4.1) satisfies this constraint and hence $\Upsilon_K(t)$ is determined. More generally, we have

Proposition 4.1. For $n \geq 8$,

$$\Upsilon_{(T_{2,-3})_{2,2n+1}}(t) = \begin{cases} -(n-1)t, & t \in [0, \frac{2}{3}] \\ 2 - (n+2)t, & t \in [\frac{2}{3}, 1] \end{cases}$$

Proof Same as the discussion above. We refer the reader to [3] for the formula of $\widehat{HFK}((T_{2,-3})_{2,2n+1})$. \square

4.2 An infinite-rank summand of topologically slice knots

Let D denote the untwisted positive whitehead double of the trefoil knot. Fix an integer $p > 2$ and let $J_n = ((D_{p,1}) \dots)_{p,1}$ denote the n -fold iterated $(p, 1)$ -cable of D for some positive integer n . Recall Corollary 1.4 states that J_n for $n = 1, 2, 3, \dots$ are linearly independent in \mathcal{C} and span an infinite-rank summand consisting of topologically slice knots. To prove this, we first establish two lemmas.

Lemma 4.2. Let ξ_n be the first singularity of $\Upsilon_{J_n}(t)$, then $\xi_n \in [\frac{1}{p^n}, \frac{2}{1+p^n}]$. In particular, $\xi_i < \xi_j$ whenever $i > j$.

Proof We first deal with the lower bound. Recall for any knot K , $\Upsilon_K(t) = -\tau(K)t$ when $t < \frac{1}{g_3(K)}$ [13]. Note $\tau(D) = g_3(D) = 1$ by [5] and hence $\tau(J_n) = p^n$ by [6]. This implies $g_3(J_n) = p^n$ since we have $p^n \leq g_4(J_n) \leq g_3(J_n) \leq p^n$. Therefore, $\xi_n \geq \frac{1}{p^n}$.

We move to establish the upper bound. Note $CFK^\infty(D) \cong CFK^\infty(T_{2,3}) \oplus A$, where A is an acyclic chain complex [7]. Therefore $\Upsilon_D(t) = \Upsilon_{T_{2,3}}(t) = |1-t|-1$. In particular, $\Upsilon_D(t) = t - 2$ when $1 \leq t \leq 2$. Apply Theorem 1.3 we get

$\Upsilon_{J_1}(t) \geq pt - 2 - (p-1)t = t - 2$ when $\frac{1}{p} \leq t \leq \frac{2}{p}$. Inductively we have $\Upsilon_{J_n}(t) \geq t - 2$ when $\frac{1}{p^n} \leq t \leq \frac{2}{p^n}$. Suppose $\xi_n > \frac{2}{1+p^n}$, then $\exists \epsilon > 0$ such that $\Upsilon_{J_n}(\frac{2}{1+p^n} + \epsilon) = -p^n(\frac{2}{1+p^n} + \epsilon) = \frac{-2p^n}{1+p^n} - \epsilon p^n < \frac{2}{1+p^n} - 2 < \frac{2}{1+p^n} + \epsilon - 2$, which is a contradiction. Therefore, $\xi_n \leq \frac{2}{1+p^n}$. \square

Let $\Delta \Upsilon'_K(t_0)$ denote the slope change of $\Upsilon_K(t)$ at t_0 , i.e. $\Delta \Upsilon'_K(t_0) = \lim_{t \searrow t_0} \Upsilon'_K(t) - \lim_{t \nearrow t_0} \Upsilon'_K(t)$. Recall in general $\frac{t_0}{2} \Delta \Upsilon'_K(t_0)$ is an integer [19]. The following lemma shows in some cases, we can determine the value of $\frac{t_0}{2} \Delta \Upsilon'_K(t_0)$.

Lemma 4.3. Let K be a knot in S^3 such that $\tau(K) \geq 0$ and let ξ be the first singularity of $\Upsilon_K(t)$. If $0 < \xi < \frac{4}{g_3(K) + \tau(K)}$, then $\frac{\xi}{2} \Delta \Upsilon'_K(\xi) = 1$.

Proof Depicting the chain complex $CFK^\infty(K)$ as lattice points in the plane, by Theorem 7.1 (3) of [13], we know there is a line of slope $1 - \frac{2}{\xi}$ containing at least two lattice points (i, j) and (i', j') supporting generators of Maslov grading 0. Since ξ is the first singularity, by Theorem 13.1 of [13] we know, say, $(i, j) = (0, \tau(K))$. So we have $\frac{j' - \tau(K)}{i'} = 1 - \frac{2}{\xi}$, which implies $0 < \xi = \frac{2i'}{i' - j' + \tau(K)} < \frac{4}{g_3(K) + \tau(K)}$. This together with the genus bound property of knot Floer homology $|i' - j'| \leq g_3(K)$ would constrain $|i'| = 1$. By Theorem 7.1 (4) of [13], $\frac{\xi}{2} \Delta \Upsilon'_K(\xi) = |i'| = 1$. \square

Proof of Corollary 1.4 Note all J_n have trivial Alexander polynomial and hence are topologically slice [2]. The linear independence follows from Lemma 4.2: suppose $\sum k_i J_{n_i} = 0$ in \mathcal{C} for some $k_i \neq 0$ and $n_1 > \dots > n_l$, since $\Upsilon_{\sum k_i J_{n_i}}(t) = \sum k_i \Upsilon_{J_{n_i}}(t)$ it possesses first singularity at ξ_{n_1} , which contradicts to $\Upsilon_{\sum k_i J_{n_i}}(t) \equiv 0$. To see they span a summand, note by Lemma 4.2 and 4.3, $\frac{\xi_n}{2} \Delta \Upsilon'_{J_n}(\xi_n) = 1$. Now one can easily see the homomorphism $\mathcal{C} \rightarrow \mathbb{Z}^\infty$ given by $[K] \mapsto (\frac{\xi_n}{2} \Delta \Upsilon'_K(\xi_n))_{n=1}^\infty$ is an isomorphism when restricted to the subgroup spanned by J_n and hence the conclusion follows. \square

Remark. One can actually replace D by any topologically slice knot K with $\tau(K) = g(K) = 1$ and even consider mixed iterated cable $((K_{p_1,1}) \dots)_{p_n,1}$. We chose J_n for the sake of an easier illustration. The linear independence of certain subfamily of mixed iterated cables of D was also observed by Feller, Park, and Ray [1].

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